State and Prove Rolle's theorem.

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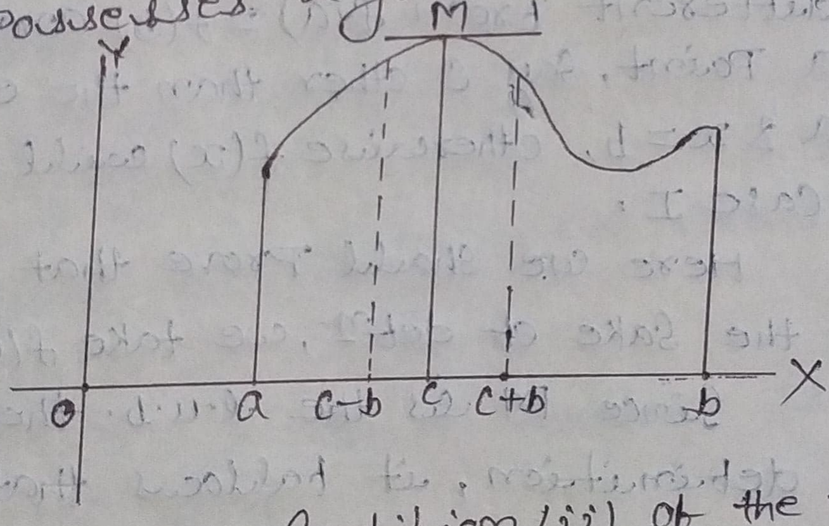
Statement:- Rolle's theorem states that,
If $f(x)$ is real valued function which is
(i) $f(x)$ is Continuous in the closed interval $[a, b]$.
(ii) $f'(x)$ exists for every Point in the open interval $]a, b[$.
(iii) $f(a) = f(b)$,
then there exists at least one Point c where $a < c < b$ i.e. $c \in]a, b[$ such that

$$f'(c) = 0.$$

Proof:- For the theorem, Ist we suppose that $f(x)$ Possesses a graph.



The Condition (ii) of the theorem Stipulates that the graph of $f(x)$ Possesses a tangent at every Point of the interval $]a, b[$ and the Conclusion of the theorem is that under the given Conditions there is a Point c Such that $a < c < b$ at which the tangent is Parallel to the x-axis.
Since $f(x)$ is Continuous in the Closed interval $[a, b]$, $f(x)$ is bounded and attains its bounds at least once in $[a, b]$.
Let its l.u.b and g.l.b be M and m respectively.

We know that $M \geqslant m$, i.e. $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$.

Case I :-

Suppose that $M = m$.

In this case,

$$f(a) = f(\alpha) = f(\beta) = f(b) = M = m \; \forall \alpha, \beta \in [a, b]$$

This means that $f(x)$ is constant in $[a, b]$.

Consequently, at point $c \in]a, b[$ we should have $f'(c) = 0$.

Case II :-

Suppose $M > m$.

In this case at least one of the bounds is different from $f(a) = f(b)$ and is attained at a point, say c other than the end points $x = a$ & $x = b$, otherwise $f(x)$ will be contt. as in case I.

Here we shall prove that $f'(c) = 0$.

For the sake of defn., we take $f(c) = M$.

Since M is the l.u.b. therefore from the definition, it follows that

$f(c+h) \leqslant f(c) = M$ and also $f(c-h) \leqslant f(c) = M$.

where, h is +ve number such that $c \pm h \in]a, b[$.

Now,

$$f(c+h) \leqslant f(c) \Rightarrow f(c+h) - f(c) \leqslant 0.$$

$$\Rightarrow \frac{f(c+h) - f(c)}{h} \leqslant 0. \quad \text{(1)}$$

and, $f(c-h) \leqslant f(c) \Rightarrow f(c-h) - f(c) \leqslant 0$,

$$\Rightarrow \frac{f(c-h) - f(c)}{-h} \geqslant 0. \quad \text{(2)}$$

Now, taking the limits of ① & ② as $h \to 0$, we have

$$\underset{h \to 0}{Lt} \cdot \frac{f(c+h) - f(c)}{h} \leq 0$$

i.e. $Rf'(c) \leq 0$ — ③

and $$\underset{h \to 0}{Lt} \frac{f(c-h) - f(c)}{-h} \geq 0.$$

i.e. $Lf'(c) \geq 0$ — ④

Now, from the Condition (ii) of the theorem we note that $f'(x)$ exist for all $x \in]a,b[$ and hence we should have,

$Rf'(c) = Lf'(c)$ since $c \in [a, b]$.

Hence, it follows from ③ & ④ we have

$Rf'(c) = 0 = Lf'(c)$.

$\therefore f'(c) = 0$.

Similarly when $f(c) = m$, we can prove that

$f'(c) = 0$.

Thus Rolle's theorem is completed.